Note on the asymptotic behavior of solutions of a system of linear ordinary differential equations

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§1. The object of this note is to investigate the asymptotic behavior of solutions of a system of linear ordinary differential equations. Our method is based on the following theorem due to M. Hukuhara.

THEOREM. *Consider a system of differential equations*

$$\frac{dx_i}{dt}=f_i(t, x_1, \cdots, x_n) \quad (i=1, \cdots, n)\ . \tag{1}$$

Let $I_1, \cdots, I_n$ be n intervals each of which has as its interior a fixed open interval I (which may be infinite), and let $t_1, \cdots, t_n$ be n arbitrarily given numbers (finite or infinite) contained in $I_1, \cdots, I_n$ respectively.

Let us assume the following conditions a)$\sim$e):

a) *$\varphi_i(t), \omega_i(t)$ $(i=1, \cdots, n)$ are continuous functions defined on I_i and $\omega_i(t)\geqq 0$.*

b) *The functions $f_i(t, x_1, \cdots, x_n)$ $(i=1, \cdots, n)$ are continuous with respect to $(x_1, \cdots, x_n)$ on a domain*

$$\mathfrak{D}_t\colon \begin{cases} |x_1-\varphi_1(t)|\leqq\omega_1(t) \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ |x_n-\varphi_n(t)|\leqq\omega_n(t) \end{cases}$$

for every fixed $t\in I$, and measurable with respect to t for every fixed $(x_1, \cdots, x_n)$.

c) *The function $f_i(t, x_1, \cdots, x_n)$ is majorized in absolute value by a function $F_i(t)$ which is summable on every closed subinterval of I_i containing t_i:*

$$|f_i(t, x_1, \cdots, x_n)|\leqq F_i(t)\ , \quad (x_1, \cdots, x_n)\in\mathfrak{D}_t \qquad (i=1, \cdots, n).$$

d) *$x_1^0, \cdots, x_n^0$ are numbers satisfying the condition*

$$|x_i^0-\varphi_i(t_i)|\leqq\omega_i(t_i) \qquad (i=1, \cdots, n)\ .$$

e) *For any system of continuous functions $x_1(t), \cdots, x_n(t)$ satisfying the condition*

$$|x_i(t)-\varphi_i(t)|\leqq\omega_i(t) \qquad (i=1, \cdots, n)\ ,$$

we have the following inequalities:

$$\left| x_i^0 + \int_{t_i}^{t} f_i(\xi, x_1(\xi), \cdots, x_n(\xi))d\xi - \varphi_i(t) \right| \leqq \omega_i(t) \qquad (i=1, \cdots, n) .$$

Then the system (1) *admits a solution* $(x_1(t), \cdots, x_n(t))$ *such that* $x_i(t)$ *is continuous on* I_i *and* $x_i(t_i)=x_i^0$.

§ 2. First we consider a system of differential equations

$$\frac{dy_i}{dt} = \lambda_i(t)y_i + \sum_{j=1}^{n} r_{ij}(t)y_j \qquad (i=1, \cdots, n) , \tag{2}$$

where the coefficients $\lambda_i(t)$ and $r_{ij}(t)$ are measurable functions on an interval $[t_0, +\infty)$.

Put

$$\Lambda_i(t) = \int_{t_0}^{t} \lambda_i(\xi)d\xi , \qquad M_i(t) = \Re(\Lambda_i(t)) ,$$

where $\Re(\lambda)$ denotes the real part of the complex number λ.

THEOREM 1. *Suppose, for a fiexd index* h $(1 \leqq h \leqq n)$, *the following conditions are satisfied.*

(A) *There exist functions* $\underline{\alpha}_i(t), \bar{\alpha}_i(t)$ *such that*

$$\underline{\alpha}_i(t) \leqq M_h(t) - M_i(t) \leqq \bar{\alpha}_i(t) \qquad (i=1, \cdots, n) .$$

(B) *The function* $R_i(t)e^{\bar{\alpha}_i(t)}$ *is summable on every finite interval* $[t_0, t]$ *for* $i=1, \cdots, m$, *and on every infinite subinterval of* $[t_0, +\infty)$ *for* $i=m+1, \cdots, n$, *where*

$$R_i(t) = \sum_{j=1}^{n} | r_{ij}(t) | + | r_{ih}(t) | \qquad (i=1, \cdots, n) .$$

(*The case of* $m=0$ *or* $m=n$ *are not excluded*).

(C) *The function* $\sigma_i(t)$ *defined by*

$$\sigma_i(t) = \begin{cases} e^{-\underline{\alpha}_i(t)} \displaystyle\int_{\tau}^{t} R_i(\xi) e^{\bar{\alpha}_i(\xi)} d\xi & (i=1, \cdots, m) , \\ e^{-\underline{\alpha}_i(t)} \displaystyle\int_{t}^{\infty} R_i(\xi) e^{\bar{\alpha}_i(\xi)} d\xi & (i=m+1, \cdots, n) , \end{cases}$$

tends to zero as $t \to +\infty$:

$$\lim_{t \to +\infty} \sigma_i(t) = 0 \qquad (i=1, \cdots, n) .$$

($\tau \geqq t_0$ *is a suitably chosen number*).

Then the system (2) *has a solution such that*

(3) $$y_1=(\delta_1^h+o(1))\exp\left(\int_\tau^t\lambda_h(\xi)d\xi\right),\ \cdots,\ y_n=(\delta_n^h+o(1))\exp\left(\int_\tau^t\lambda_h(\xi)d\xi\right),$$

where δ_j^h *denotes the Kronecker's symbol.*

Proof. By the change of variables

$$y_i=e^{\Lambda_i(t)}z_i \qquad (i=1,\cdots,n),$$

the system (2) is transformed into a new system

(4) $$\frac{dz_i}{dt}=\sum_{j=1}^n r_{ij}(t)e^{\Lambda_j(t)-\Lambda_i(t)}z_j \qquad (i=1,\cdots,n).$$

In order to prove the theorem it is sufficient to show that the system (4) has a solution such that

(5) $$z_1=(\delta_1^h+o(1))e^{\Lambda_h(t)-\Lambda_1(t)},\ \cdots,\ z_n=(\delta_n^h+o(1))e^{\Lambda_h(t)-\Lambda_n(t)}.$$

Let us put

$$\left.\begin{array}{l} f_i(t,z_1,\cdots,z_n)=\sum_{j=1}^n r_{ij}(t)e^{\Lambda_j(t)-\Lambda_i(t)}z_j\ , \\ \varphi_i(t)=\delta_i^h\ ,\quad \omega_i(t)=e^{\underline{\alpha}_i(t)}\sigma_i(t)\ ,\quad z_i^0=\delta_i^h \end{array}\right\} \qquad (i=1,\cdots,n),$$

and

$$t_i=\begin{cases}\tau & (i=1,\cdots,m),\\ +\infty & (i=m+1,\cdots,n).\end{cases}$$

For any system of continuous functions $z_1(t),\cdots,z_n(t)$ satisfying the condition

(6) $$|z_i(t)-\varphi_i(t)|\leqq\omega_i(t) \qquad (t\geqq\tau,\ i=1,\cdots,n),$$

we have

$$|f_i(t,z_1(t),\cdots,z_n(t))|\leqq\sum_{j=1}^n|r_{ij}(t)|e^{M_j(t)-M_i(t)}\omega_j(t)+|r_{ih}(t)|e^{M_h(t)-M_i(t)}$$
$$=e^{M_h(t)-M_i(t)}\left\{\sum_{j=1}^n|r_{ij}(t)|e^{\underline{\alpha}_j(t)-(M_h(t)-M_j(t))}\sigma_j(t)+|r_{ih}(t)|\right\} \quad (i=1,\cdots,n).$$

According to the condition (C) we can presuppose without loss of generality that

$$0\leqq\sigma_i(t)\leqq1 \text{ for } t\geqq\tau \qquad (i=1,\cdots,n).$$

Thus, using the condition (A), we have

$$|f_i(t,z_1(t),\cdots,z_n(t))|\leqq R_i(t)e^{\overline{\alpha}_i(t)} \qquad (t\geqq\tau,\ i=1,\cdots,n),$$

and finally

$$\left|z_i^0+\int_{t_i}^{t} f_i(\xi, z_1(\xi), \cdots, z_n(\xi))d\xi-\varphi_i(t)\right| \leqq \omega_i(t) \qquad (i=1, \cdots, n) ,$$

where $(z_1(t), \cdots, z_n(t))$ is any system of continuous functions satisfying the condition (6).

Therefore, by the theorem of Hukuhara, the system (4) has a solution $(z_1(t), \cdots, z_n(t))$ which satisfies the following conditions:

$$|z_i(t)-\varphi_i(t)| \leqq \omega_i(t) , \quad z_i(t_i)=z_i^0 \qquad (t \geqq \tau, i=1, \cdots, n) ,$$

i.e.

$$(7) \qquad \begin{aligned} |z_i(t)-\delta_i^h| &\leqq \omega_i(t) && (t \geqq \tau, i=1, \cdots, n) , \\ z_i(t_i) &= \delta_i^h && (i=1, \cdots, n) . \end{aligned}$$

Multiplying (7) by $e^{-(M_h(t)-M_i(t))}$, we obtain the inequalities

$$|z_i(t)e^{-(\Lambda_h(t)-\Lambda_i(t))}-\delta_i^h e^{-(\Lambda_h(t)-\Lambda_i(t))}| \leqq e^{-(M_h(t)-M_i(t))}\omega_i(t),$$

from which, by the condition (A), it follows that

$$|z_i(t)e^{-(\Lambda_h(t)-\Lambda_i(t))}-\delta_i^h| \leqq e^{-\underline{\alpha}_i(t)}\omega_i(t)=\sigma_i(t) \qquad (i=1, \cdots, n) .$$

In virtue of the condition (C) the righthand sides of these inequalities tend to zero as $t \to +\infty$, and so the system $(z_1(t), \cdots, z_n(t))$ is easily seen to be nothing but the solution (5).

Thus, Theorem 1 is completely proved.

§ 3. In this section we prove a corollary of Theorem 1. We preserve the previous definitions of the functions $M_i(t)$, $R_i(t)$, the index h being fixed.

COROLLARY. *Let us assume the following conditions* a)~e):

a) *There exist functions* $\underline{\alpha}_i(t)$, $\bar{\alpha}_i(t)$ *such that*

$$\underline{\alpha}_i(t) \leqq M_h(t)-M_i(t) \leqq \bar{\alpha}_i(t) \qquad (i=1, \cdots, n) .$$

b) *The functions* $\underline{\alpha}_i(t)$, $\bar{\alpha}_i(t)$ *are non-decreasing for* $i=1, \cdots, m$, *and*

$$\lim_{t\to+\infty} \underline{\alpha}_i(t)=+\infty \qquad (i=1, \cdots, m) .$$

c) *The functions* $\underline{\alpha}_i(t)$, $\bar{\alpha}_i(t)$ *are non-increasing for* $i=m+1, \cdots, n$.

d) *There exists a number* τ *such that*

$$\bar{\alpha}_i(t)-\underline{\alpha}_i(t) \leqq K_i+\beta_i(t) \log t \quad \textit{for } t \geqq \tau \qquad (i=1, \cdots, n) ,$$

where K_i *is a non-negative number, and* $\beta_i(t)$ *a non-negative measurable function.*

e) $$\int_{\tau}^{\infty} R_i(\xi)\xi^{\beta_i(\xi)}d\xi < +\infty \qquad (i=1, \cdots, n)\,.$$

Then the system (2) *has a solution* (3).

Proof. It is readily seen from the condition d) that

$$R_i(t)e^{\bar{\alpha}_i(t)} \leqq e^{K_i}R_i(t)e^{\underline{\alpha}_i(t)}t^{\beta_i(t)} \qquad (i=1, \cdots, n)\,.$$

Therefore, by the condition e), the function $R_i(t)e^{\bar{\alpha}_i(t)}$ is summable on every interval $[\tau, t]\,(t<+\infty)$ for $i=1, \cdots, m$, and on every infinite subinterval of $[\tau, +\infty)$ for $i=m+1, \cdots, n$, since $\underline{\alpha}_i(t)$ is bounded from above.

Thus it remains only to prove that the functions

$$\sigma_i(t)=\begin{cases} e^{-\underline{\alpha}_i(t)}\displaystyle\int_{\tau}^{t} R_i(\xi)e^{\bar{\alpha}_i(\xi)}d\xi & (i=1, \cdots, m)\,, \\ e^{-\underline{\alpha}_i(t)}\displaystyle\int_{t}^{\infty} R_i(\xi)e^{\bar{\alpha}_i(\xi)}d\xi & (i=m+1, \cdots, n)\,, \end{cases}$$

satisfy the condition C) of Theorem 1.

For any $\varepsilon>0$ there exists, by virtue of the condition e), a number t^* $(t^*\geqq\tau)$ such that

$$e^{K_i}\int_{t^*}^{\infty} R_i(\xi)\xi^{\beta_i(\xi)}d\xi<\frac{\varepsilon}{2} \qquad (i=1, \cdots, n)\,.$$

Let us put

$$e^{-\underline{\alpha}_i(t)}\int_{\tau}^{t} R_i(\xi)e^{\bar{\alpha}_i(\xi)}d\xi=\mathrm{I}+\mathrm{II} \qquad (i=1, \cdots, m)\,,$$

where

$$\mathrm{I}=e^{-\underline{\alpha}_i(t)}\int_{\tau}^{t^*} R_i(\xi)e^{\bar{\alpha}_i(\xi)}d\xi \ \ (t\geqq t^*)\,, \qquad \mathrm{II}=e^{-\underline{\alpha}_i(t)}\int_{t^*}^{t} R_i(\xi)e^{\bar{\alpha}_i(\xi)}d\xi \ \ (t\geqq t^*)\,.$$

Since $\lim\limits_{t\to+\infty}\underline{\alpha}_i(t)=+\infty$ $(i=1, \cdots, m)$, it is evident that

$$\mathrm{I}<\frac{\varepsilon}{2} \quad \text{for } t\geqq t'\,,$$

where t' is a sufficiently large number.

On the other hand

$$\mathrm{II}\leqq e^{-\underline{\alpha}_i(t)}\int_{t^*}^{t} R_i(\xi)e^{\underline{\alpha}_i(\xi)+K_i+\beta_i(\xi)\log\xi}d\xi\leqq e^{-\underline{\alpha}_i(t)+K_i}\int_{t^*}^{t} R_i(\xi)e^{\underline{\alpha}_i(t)}\xi^{\beta_i(\xi)}d\xi$$
$$=e^{K_i}\int_{t^*}^{t} R_i(\xi)\xi^{\beta_i(\xi)}d\xi<\frac{\varepsilon}{2} \qquad (i=1, \cdots, m)\,.$$

Hence

$$\lim_{t\to+\infty} \sigma_i(t)=0 \qquad (i=1, \cdots, m) .$$

From the fact that $-\underline{\alpha}_i(t)\,(i=m+1, \cdots, n)$ are non-decreasing functions, and from the condition d) it follows that

$$e^{-\underline{\alpha}_i(t)}\int_t^\infty R_i(\xi)e^{\overline{\alpha}_i(\xi)}d\xi \leqq \int_t^\infty R_i(\xi)e^{\overline{\alpha}_i(\xi)-\underline{\alpha}_i(\xi)}d\xi$$
$$\leqq e^{K_i}\int_t^\infty R_i(\xi)\xi^{\beta_i(\xi)}d\xi \longrightarrow 0 \qquad \text{as } t\longrightarrow+\infty ,$$

that is

$$\lim_{t\to+\infty} \sigma_i(t)=0 \qquad (i=m+1, \cdots, n) .$$

Thus our assertion is proved.

§ 4. Here we investigate a more general case, where a system of equations

$$(8) \qquad \frac{dx_i}{dt} = \sum_{j=1}^n (a_{ij}+b_{ij}(t)+c_{ij}(t))x_j \qquad (i=1, \cdots, n)$$

is to be considered.

THEOREM 2. *Let $A=(a_{ij})$ be a constant matrix with characteristic roots $\mu_j, j=1, \cdots, n$, all of which are distinct, and $b_{ij}(t)$ be functions differentiable on $[t_0, +\infty)$ with the additional property:*

$$\lim_{t\to+\infty} b_{ij}(t)=0 \qquad (i, j=1, \cdots, n) ,$$

and $c_{ij}(t)$ be measurable on $[t_0, +\infty)$.
Denote the characteristic roots of the matrix $(a_{ij}+b_{ij}(t))$ by $\lambda_1(t), \cdots, \lambda_n(t)$. We may assume, by reordering if necessary, that $\lim_{t\to+\infty}\lambda_j(t)=\mu_j$.

Let us put

$$M_i(t)=\int_{t_0}^t \Re(\lambda_i(\xi))d\xi \qquad (i=1, \cdots, n) ,$$

and assume, for a fixed index h $(1\leqq h\leqq n)$, the following conditions:

(A) *There exist functions $\underline{\alpha}_i(t), \overline{\alpha}_i(t)$ such that*

$$\underline{\alpha}_i(t)\leqq M_h(t)-M_i(t)\leqq\overline{\alpha}_i(t) \qquad (i=1, \cdots, n) .$$

(B) *The functions $\underline{\alpha}_i(t), \overline{\alpha}_i(t)$ are non-decreasing for $i=1, \cdots, m$, and*

$$\lim_{t\to+\infty} \underline{\alpha}_i(t)=+\infty .$$

(C) *The functions* $\underline{\alpha}_i(t)$, $\bar{\alpha}_i(t)$ *are non-increasing for* $i=m+1, \cdots, n$.

(D) *There exists a number* τ *such that*

$$\bar{\alpha}_i(t)-\underline{\alpha}_i(t)\leqq K+\beta(t)\log t \quad \textit{for } t\geqq\tau \qquad (i=1, \cdots, n),$$

where K *is a non-negative number, and* $\beta(t)$ *a non-negative measurable function.*

(E)

$$\int_\tau^\infty |b'_{ij}(\xi)|\,\xi^{\beta(\xi)}d\xi<\infty, \quad \int_\tau^\infty |c_{ij}(\xi)|\,\xi^{\beta(\xi)}d\xi<\infty \qquad (i, j=1, \cdots, n)\,.$$

Then, there is a solution $(x_1, \cdots, x_n)$ *of the system* (8) *such that*

$$x_1=(s_1+o(1))\exp\left(\int_\tau^t\lambda_h(\xi)d\xi\right), \cdots, x_n=(s_n+o(1))\exp\left(\int_\tau^t\lambda_h(\xi)d\xi\right),$$

where $(s_1, \cdots, s_n)$ *is a characteristic vector of the matrix* A *associated with* μ_h.

Proof. First we construct the matrix $P(t)=(p_{ij}(t))$ so that the change of variables $\mathbf{x}=P(t)\mathbf{y}$ reduces the system (8) to the system of the type (2). After the reduction, the corollary of Theorem 1 will be applied to the new system and thus the proof of our assertion will be completed.

Since all the characteristic roots of the matrix A are distinct, there exists a constant non-singular matrix $S=(s_{ij})$ such that

$$S^{-1}AS=D\,,$$

where D is a diagonal matrix:

$$D=\begin{pmatrix}\mu_1 & & & 0\\ & \mu_2 & & \\ & & \ddots & \\ 0 & & & \mu_n\end{pmatrix}.$$

Define the matrix $F(t)=(f_{ij}(t))$ by

$$F(t)=S^{-1}B(t)S\,,$$

where $B(t)$ stands for the matrix having $b_{ij}(t)$ as the element in the i-th row and j-th column. Let us define the matrix $M(\lambda, t)$ as follows:

$$M(\lambda, t)=D+F(t)-\lambda E\,.$$

Denoting the cofactor of the element in the j-th row and i-th column of $M(\lambda, t)$ by $w_{ji}(\lambda, t)$, we will put

$$\tilde{p}_{ij}(t)=\frac{w_{ji}(\lambda_j(t), t)}{\prod\limits_{s\neq j}(\mu_s-\mu_j)}\,.$$

Since $F(t)\to 0$ as $t\to+\infty$, it is easily seen that $w_{ji}(\lambda_j(t), t)$ tends to the cofactor of the element in the j-th row and i-th column of the matrix $D-\mu_j E$ as $t\to+\infty$:

$$\lim_{t\to+\infty} w_{ji}(\lambda_j(t), t)=\begin{cases}0 & (i\neq j)\ ,\\ \prod_{s\neq j}^{n}(\mu_s-\mu_j) & (i=j)\ ,\end{cases}$$

which implies

$$\lim_{t\to+\infty}\tilde{p}_{ij}(t)=\delta^i_j\ .$$

Let the matrix with elements $\tilde{p}_{ij}(t)$ be $\tilde{P}(t)$. Since the characteristic roots of the matrix $D+F(t)$ coincide with those of the matrix $A+B(t)$, $\det M(\lambda_j(t), t)=0$ and so it follows that

$$\begin{aligned}&\sum_{k=1}^{n}(\mu_i\delta^i_k+f_{ik}(t)-\lambda_j(t)\delta^i_k)\tilde{p}_{kj}(t)\\ &=\frac{1}{\prod_{s\neq j}^{n}(\mu_s-\mu_j)}\sum_{k=1}^{n}(\mu_i\delta^i_k+f_{ik}(t)-\lambda_j(t)\delta^i_k)w_{jk}(\lambda_j(t), t)\\ &=0 \qquad\qquad (i=1,\cdots,n)\ ,\end{aligned}$$

i.e.
$$\sum_{k=1}^{n}(\mu_i\delta^i_k+f_{ik}(t))\tilde{p}_{kj}(t)=\lambda_j(t)\tilde{p}_{ij}(t)\qquad (i=1,\cdots,n)\ ,$$

and finally we have

$$(D+F(t))\tilde{P}(t)=\tilde{P}(t)\Lambda(t)\ ,$$

where $\Lambda(t)$ is a diagonal matrix:

$$\Lambda(t)=\begin{pmatrix}\lambda_1(t) & & & 0\\ & \lambda_2(t) & & \\ & & \ddots & \\ 0 & & & \lambda_n(t)\end{pmatrix}.$$

From the fact that $\tilde{P}(\infty)=E$, it can be seen that there exists an inverse matrix $\tilde{P}^{-1}(t)$ of $\tilde{P}(t)$ for sufficiently large t. Therefore

$$\tilde{P}^{-1}(t)(D+F(t))\tilde{P}(t)=\Lambda(t)\ .$$

If we define the matrix $P(t)=(p_{ij}(t))$ by

$$P(t)=S\tilde{P}(t)\ ,$$

then for sufficiently large t

$$\begin{aligned}P^{-1}(t)(A+B(t))P(t)&=P^{-1}(t)S^{-1}(A+B(t))S\tilde{P}(t)\\ &=\tilde{P}^{-1}(t)(S^{-1}AS+S^{-1}B(t)S)\tilde{P}(t)=\tilde{P}^{-1}(t)(D+F(t))\tilde{P}(t)=\Lambda(t)\ .\end{aligned}$$

Since every element of matrix $F(t)$ is linear in the elements of $B(t)$, it follows from the condition (E) that

$$\int_{\tau}^{\infty} |f'_{ij}(\xi)| \xi^{\beta(\xi)} d\xi < \infty \qquad (i, j=1, \cdots, n) .$$

Let us denote $\det M(\lambda, t)$ by $G(\lambda, t)$. Then $G(\lambda_j(t), t)=0$ and

$$\frac{\partial G}{\partial \lambda}(\lambda_j(t), t)\lambda'_j(t) + \frac{\partial G}{\partial t}(\lambda_j(t), t)=0 .$$

Since the term $(\partial G/\partial t)(\lambda_j(t), t)$ is linear homogeneous in $f'_{ij}(t)$, with all the elements of $M(\lambda_j(t), t)$ being bounded for sufficiently large t, it follows that

$$\int_{\tau'}^{\infty} \left| \frac{\partial G}{\partial \xi}(\lambda_j(\xi), \xi) \right| \xi^{\beta(\xi)} d\xi < \infty , \tag{9}$$

where $\tau'(\geqq \tau)$ is a sufficiently large number. Without loss of generality we can assume $\tau'=\tau$ for the rest of our discussion. The term $(\partial G/\partial \lambda)(\lambda_j(t), t)$ tends to a non-vanishing limit as $t \to +\infty$ since the characteristic roots of D are distinct. So, by virtue of (9), we have

$$\int_{\tau}^{\infty} |\lambda'_j(\xi)| \xi^{\beta(\xi)} d\xi < \infty \qquad (j=1, \cdots, n). \tag{10}$$

Taking account of the fact that $\tilde{p}'_{ij}(t)$ is linear homogeneous in the elements $f'_{ks}(t)$ and $\lambda'_j(t)$, and using the boundedness of all the elements of $M(\lambda_j(t), t)$ for sufficiently large t, we can deduce from the former condition of (E) and (10) the inequality

$$\int_{\tau}^{\infty} |\tilde{p}'_{ij}(\xi)| \xi^{\beta(\xi)} d\xi < \infty .$$

Thus clearly

$$\int_{\tau}^{\infty} |p'_{ij}(\xi)| \xi^{\beta(\xi)} d\xi < \infty \qquad (i, j=1, \cdots, n) .$$

Let us transform the system (8) by the change of variables

$$x_i = \sum_{j=1}^{n} p_{ij}(t) y_j \qquad (i=1, \cdots, n) ,$$

into a system

$$\frac{dy_j}{dt} = \lambda_j(t) y_j + \sum_{k=1}^{n} r_{jk}(t) y_k \qquad (j=1, \cdots, n) , \tag{11}$$

where

$$(r_{jk}(t))=(p_{jk}(t))^{-1}(c_{jk}(t))(p_{jk}(t))-(p_{jk}(t))^{-1}(p'_{jk}(t)) .$$

Since the element $r_{jk}(t)$ is linear homogeneous in the elements $c_{is}(t)$ and $p'_{ik}(t)$, with all the elements of $P(t)$ and $P^{-1}(t)$ being bounded for sufficiently large t, we obtain the inequalities

$$\int_{\tau}^{\infty} | r_{ij}(\xi) | \xi^{\beta(\xi)} d\xi < \infty \qquad (i, j=1, \cdots, n) ,$$

which imply the following ones:

$$\int_{\tau}^{\infty} R_i(\xi)\xi^{\beta(\xi)} d\xi < \infty \qquad (i=1, \cdots, n) ,$$

where $R_i(t)= \sum_{j=1}^{n} | r_{ij}(t) | + | r_{ih}(t) |$.

Applying the corollary of Theorem 1 to the system (11), we see that the system (11) admits a solution

$$y_j=(\delta_j^h+o(1)) \exp\left(\int_{\tau}^{t}\lambda_h(\xi)d\xi\right) \qquad (j=1, \cdots, n) ,$$

and consequently the system (8) has as a solution

$$x_i= \sum_{j=1}^{n} p_{ij}(t)(\delta_j^h+o(1)) \exp\left(\int_{\tau}^{t}\lambda_h(\xi)d\xi\right) \qquad (i=1, \cdots, n) .$$

Because $P(t)=S\tilde{P}(t)$, clearly $\lim_{t\to+\infty} p_{ij}(t)=s_{ij}$, where s_{ij} stands for the element in the i-th row and j-th column of the matrix S. Therefore it follows that

$$\lim_{t\to+\infty} \exp\left(-\int_{\tau}^{t}\lambda_h(\xi)d\xi\right)x_i=s_{ih} \qquad (i=1, \cdots, n) ,$$

i.e.

$$x_i=(s_{ih}+o(1)) \exp\left(\int_{\tau}^{t}\lambda_h(\xi)d\xi\right) \qquad (i=1, \cdots, n) .$$

The vector $(s_{1h}, \cdots, s_{nh})$ is a characteristic vector of the matrix A associated with μ_h, since $AS=SD$. Considering $(s_{1h}, \cdots, s_{nh})$ obtained above as our $(s_1, \cdots, s_n)$ we complete the proof of Theorem 2.

§ 5. In this section we give a proof of a theorem of N. Levinson as a direct consequence of Theorem 2.

COROLLARY. *Consider the system of equations* (8). *Let* $A=(a_{ij})$, μ_j, $b_{ij}(t)$, *and* $\lambda_j(t)$ *be as in the first two sentences of Theorem* 2.

Suppose, for a fixed index h $(1\leqq h\leqq n)$, *that*

$$\Re(\lambda_h(t)-\lambda_i(t)) \qquad (i=1, \cdots, n)$$

satisfy the following conditions (a), (b).

$$\text{(a)} \qquad \begin{cases} \lim_{t\to+\infty} \int_{t_0}^{t} \Re(\lambda_h(\xi)-\lambda_i(\xi))d\xi = +\infty \ , \quad \textit{and} \\ \int_{t_1}^{t_2} \Re(\lambda_h(\xi)-\lambda_i(\xi))d\xi \geqq -C \quad (t_2 \geqq t_1 \geqq t_0) \end{cases} \qquad (i=1, \cdots, m) \ .$$

$$\text{(b)} \qquad \int_{t_1}^{t_2} \Re(\lambda_h(\xi)-\lambda_i(\xi))d\xi \leqq C \quad (t_2 \geqq t_1 \geqq t_0) \qquad (i=m+1, \cdots, n) \ .$$

(C is a positive number). Then

$$(*) \qquad \int_{t_0}^{\infty} |b'_{ij}(\xi)|\, d\xi < \infty \ , \quad \int_{t_0}^{\infty} |c_{ij}(\xi)|\, d\xi < \infty \qquad (i, j=1, \cdots, n)$$

is a sufficient condition that the system (8) *have as a solution*

$$x_1=(s_1+o(1)) \exp\left(\int_{\tau}^{t} \lambda_h(\xi)d\xi\right), \cdots, x_n=(s_n+o(1)) \exp\left(\int_{\tau}^{t} \lambda_h(\xi)d\xi\right) ,$$

where $(s_1, \cdots, s_n)$ is a characteristic vector of A associated with μ_h.

Proof. Using our previous notation, we have

$$\int_{t_0}^{t} \Re(\lambda_h(\xi)-\lambda_i(\xi))d\xi = M_h(t)-M_i(t) \ .$$

Let us define the functions $\underline{\alpha}_i(t), \bar{\alpha}_i(t)$ as follows:

$$\bar{\alpha}_i(t) = \begin{cases} \sup_{t_0 \leqq t' \leqq t} (M_h(t')-M_i(t')) & (i=1, \cdots, m) \ , \\ \sup_{t' \geqq t} (M_h(t')-M_i(t')) & (i=m+1, \cdots, n) \ , \end{cases}$$

$$\underline{\alpha}_i(t) = \begin{cases} \inf_{t' \geqq t} (M_h(t')-M_i(t')) & (i=1, \cdots, m) \ , \\ \inf_{t_0 \leqq t' \leqq t} (M_h(t')-M_i(t')) & (i=m+1, \cdots, n) \ . \end{cases}$$

Then by virtue of the conditions (a), (b) it is easily seen that

$$\underline{\alpha}_i(t) \leqq M_h(t)-M_i(t) \leqq \bar{\alpha}_i(t) \ ,$$
$$\bar{\alpha}_i(t)-\underline{\alpha}_i(t) \leqq C \qquad (t \geqq t_0, i=1, \cdots, n) \ ,$$

and clearly the functions $\underline{\alpha}_i(t), \bar{\alpha}_i(t)$ satisfy the conditions (A), (B), (C) and (D) of Theorem 2 with $K=C, \beta(t)\equiv 0$. Since $\beta(t)\equiv 0$, the condition (E) of Theorem 2 is necessarily satisfied in the present case by virtue of the condition (*). Thus our assertion is an immediate consequence of Theorem 2.

Remark. The inequalities

$$\bar{\alpha}_i(t)-\underline{\alpha}_i(t) \leqq C \qquad (i=1, \cdots, n)$$

could also be obtained by putting $K=0, \beta(t)=C/\log t$. Then the corresponding conditions become

$$\int_{t_0}^{\infty} | b'_{ij}(\xi) | \xi^{\sigma/\log\xi} d\xi < \infty , \quad \int_{t_0}^{\infty} | c_{ij}(\xi) | \xi^{\sigma/\log\xi} d\xi < \infty \qquad (i, j = 1, \cdots, n) ,$$

which are clearly equivalent to the conditions (*), since $\xi^{\sigma/\log\xi} = e^{\sigma}$.

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